ALGEBRA ORAL

(1) Let k be a field and $B \subset \mathrm{GL}_2(k)$ the subgroup of upper triangles. Describe all $B \times B$-orbits in $\mathrm{GL}_2(k)$, where $B \times B$ acts on $\mathrm{GL}_2(k)$ by
$(b_1, b_2) \cdot g = b_1 g b_2^{-1},\ \forall (b_1, b_2) \in B \times B, g \in \mathrm{GL}_2(k)$.

(2) Let k be a field and $A, B \in M_{n\times n}(k)$. Prove that $\det(I - AB) = \det(I - BA)$.

(3) Find $\mathbb{Q}$-algebras R such that in $R[X, Y]$, $X^2 + Y^2 = (aX + bY)^2$ for some $a, b \in R$.